

Social Structure and Opinion Formation

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Abstract

We present a dynamical theory of opinion formation that takes explicitly into account the structure of the social network in which individuals are embedded. This is the case that arises in situations as varied as the adoption of new technologies and political views. The theory predicts the evolution of a set of opinions through a social network and establishes the existence of a martingale property, i.e. that the expected weighted fraction of the population that holds a given opinion is constant in time. The distribution of the fraction stabilizes only after a long time that diverges with the system size. This coexistence of opinions within a social network is in agreement with the often observed locality effect, in which an opinion or a fad is localized to given groups without infecting the *whole* society. We verified these predictions, as well as those concerning the fragility of opinions and the importance of highly connected individuals in opinion formation, by performing computer experiments on a number of social networks.

1 Introduction

Most people hold opinions about a myriad topics, from politics and entertainment to health, new products and the lives of others. These opinions can be either the result of serious reflection or, as is often the case when information is hard to process or obtain, formed through interactions with others that hold views on given issues. This reliance on others to form opinions lies at the heart of advertising through social cues, efforts to make people aware of societal and health related issues, fads that sweep social groups and organizations, and attempts at capturing the votes and minds of people in election years. Because of our dependence on others to shape our views of the world, an understanding of opinion formation requires an examination of the interplay between the structure of the social network in which individuals are embedded and the interactions that take place within it.

This explains the vast efforts that both the commercial and the public sectors devote to uncovering such interplay and the mechanisms they deploy to affect the formation of favorable and unfavorable opinions about any imaginable topic. Other problems that often depend on opinion formation are the diffusion of innovations and technology adoption, since other people's views do influence the purchase of a product or the acquisition of a new technology [1, 33, 2, 27].

More recently, both the emergence of email and global access to information through the web has started to change the discourse in civil society [8, 28, 31, 32, 41]. and made it even easier to propagate points of view and misleading facts through vast numbers of people; views which are surprisingly accepted and transmitted on to others without much critical examination. In the academic arena there exist several models of opinion formation that take into account some factors while leaving out others (for one of the earliest see [12]).

In economics, information cascades were proposed to explain uniformity in social behavior [4, 5], as well as its fragility. This approach assumes that there is a linear sequence of Bayesian individuals that can observe the choices of others in front of them before making their own decisions as to which opinion to choose. Besides the notorious problems with assuming Bayesian decision makers [6] this theory makes unrealistic assumptions, such as assuming a sequence of synchronous decisions that does not take into account the social network, or locality of contacts, that people have. More problematic is the prediction that given an initial set of possible opinions, the information cascade will lead to one opinion eventually becoming pervasive,

which contradicts the common observation that conformity throughout a society tends to be localized in subgroups rather than widespread. Other approaches to opinion formation have dealt with either theoretical models or computer simulations. On the theory side a number of dynamical models that have been proposed are based on analogies to discrete state magnetic systems placed either on two dimensional lattices [17, 19, 11]. While none of them takes into account the social structure in shaping opinion formation, in the case of computer simulations, models that have a continuum of possible opinions or very large number of opinions can sometimes yield asymptotic states that are non-uniform [9, 23, 35, 36, 16], partly due to the many choices of opinions. However, all models with a binary choice of opinions lead to widespread dominance [18, 19, 38, 20, 26, 39, 40], once again in disagreement with observations.

In this paper we propose a theory of opinion formation that explicitly takes into account the structure of the social network in which individuals are embedded. The theory assumes asynchronous choices by individuals among two or three opinions and it predicts the time evolution of the set of opinions from any arbitrary initial condition. We show that on very general network structures a martingale property ensues, i.e. that the expected weighted fraction of the population that holds a given opinion is constant in time. By weighted fraction we mean the fraction of individuals holding a given opinion, averaged over their social connectivity. Most importantly, the distribution of this weighted fraction of opinions does not stabilize to zero or one for a considerable long time that diverges with the network size. This coexistence of opinions within a social network is in agreement with the often observed locality effect, in which an opinion or a fad is localized to given groups without infecting the *whole* society.

Our theory further predicts that a relatively small number of individuals with high social rank can have a larger effect on opinion formation than individuals with low rank. By high rank we mean people with a large number of social connections. This also explains the fragility phenomenon, whereby an opinion that seems to be held by a rather large group of people can become nearly extinct in a very short time, a mechanism that is at the heart of fads. These predictions were verified by computer experiments and extended to the case when some individuals hold fixed opinions throughout the dynamical process. Furthermore, we dealt with the case of information asymmetries, which are characterized by the fact that some individuals are often influenced by other people's opinions while being unable to reciprocate and change their counterpart's views. In the following sections we describe the dynamical model and proceed to solve it analytically. We then extend it

to several interesting cases (fixed opinions and information asymmetries) and then present the results of computer simulations that confirm the theoretical predictions. A concluding section summarizes our results and discusses their implications to opinion formation and possible future research.

2 Two opinions within a social network

2.1 Description of the model

In our theory we represent a social network as a random graph with a certain degree distribution p_k . The nodes of this graph correspond to people and the edges represent their social connection. For the sake of generality, we will assume that the degree of each node is drawn independently from an arbitrary distribution p_k , so that any two graphs with the same degree sequence are equally likely. One can imagine that every node with degree k has k edges “sticking out”, and these edges are matched off pairwise in a random way. Therefore, the probability that two linked nodes have degrees j and k in order is proportional to jp_jkp_k . This point is made clear in Section II of [30]. This description of a society is supported by a number of empirical studies [3] that have shown that many social networks are scale free, i.e. the number of ties or links per node follows a power law. By this it is meant that the graph representing the social interactions is self-similar, so that any particular part of the graph is like a sample of the whole network.

Instead of assuming a random graph, we can alternatively assume a large deterministic graph with a degree distribution p_k , in which the fraction of linked ordered pairs with degrees j and k is proportional to jp_jkp_k . We say such a network is *uniformly connected*. All our results in this paper hold under this assumption.

In the following discussion we also assume that the structure of the social network changes over time scales that are much slower than opinion formation, so that for all practical purposes the graph structure can be considered static over the time that opinions form. We use the terms “black” and “white” to denote the binary opinions available to each person, who is represented by a node. A person (node) is either of the black or of the white opinion. We then assume that starting from an initial color distribution, people asynchronously update their opinions at a rate λ . That is, during any time interval dt , each node updates its color (makes a decision as to which opinion to hold) with probability λdt , based on the colors of its neighbors. Specifically, if a given person or node has b black neighbors and w white neighbors, then the probability that its new color is going to be black is

Table 1: Symbols and their meanings

n	total number of nodes
n_k	number of nodes with degree k
$p_k = n_k/n$	the degree distribution
m	total number of black nodes
m_k	number of black nodes with degree k
$q = m/n$	fraction of black nodes
$q_k = m_k/n_k$	fraction of black nodes in all degree- k nodes

$b/(b+w)$. This is equivalent to assuming that each time a person randomly chooses one of its neighbors and sets its new color to be the same as that neighbor. Note that when we say that a person or node “updates”, we do not mean “changes”. It is completely possible that a node remains the same color after the update.

This specific updating rule is known in the social dynamics literature as the *voter model*, whose dynamics has been particularly studied on heterogeneous networks [34], small-world networks [7] and BA networks [37]. Of course it is oversimplified as in reality one seldom changes his opinion constantly, and we will relax this assumption later in Section 4. In this paper we will study the dynamics of the voter model on networks with most general topologies. We will determine how opinions spread throughout the as a function of time on a given social structure. As we show, which opinion (or color) will prevail is not obvious, as well as how the ratio of black-to-white changes with time? We will first consider the case where once the opinion formation starts, no new sources of opinions enter the social network. We will then relax this assumption by allowing for new opinions to enter into a social network as time evolves. Throughout this paper we will use the following symbols and their meaning, which are listed in Table 1.

2.2 The dynamics of opinion formation

Consider a specific update that happens at some time t . Let A be the person or node that updates, and let k be its degree. Because all n nodes update their colors asynchronously and independently of each other at the same rate, everyone has the same chance to be observed updating at time t . Thus the degree distribution of A is just the degree distribution of a randomly chosen node, or p_k . During the update, A randomly copies the color from

one of its neighbors, which we will call B . We calculate the change of m_k due to this specific update. There are three cases:

1. A is white and B is black. A updates its color to black and consequently increases m_k by 1.
2. A is black and B is white. A updates its color to white and consequently decreases m_k by 1.
3. A and B have the same color. In this case m_k does not change.

Given A 's degree k , the probability that A is black or white before the update is simply q_k or $1 - q_k$ by definition. To calculate the black probability of B we need to know its degree distribution first, which in our case is not p_k . This is because A being a randomly chosen node is more likely to be a neighbor of a high degree node than a low degree node. Specifically, under the uniform connection assumption, the probability that B has degree j is proportional to jp_j [15]. Conditioning on the event that B has degree j , the black probability of B is again simply q_j . Thus, the probability that the update changes a degree- k node from white to black (case 1) is given by

$$P_{w \rightarrow b}(k) = p_k(1 - q_k) \frac{\sum_j jp_j q_j}{\sum_j jp_j}. \quad (1)$$

Similarly, the probability that the update changes a degree- k node from black to white (case 2) is given by

$$P_{b \rightarrow w}(k) = p_k q_k \frac{\sum_j jp_j (1 - q_j)}{\sum_j jp_j} = p_k q_k \left(1 - \frac{\sum_j jp_j q_j}{\sum_j jp_j} \right). \quad (2)$$

If we define

$$\langle q \rangle = \frac{\sum_j jp_j q_j}{\sum_j jp_j} \quad (3)$$

to be a weighted average over all q_k 's, then the two probabilities can be written as

$$P_{w \rightarrow b}(k) = p_k(1 - q_k)\langle q \rangle, \quad (4)$$

$$P_{b \rightarrow w}(k) = p_k q_k(1 - \langle q \rangle). \quad (5)$$

This gives us the increment of m_k due to a particular update:

$$\Delta m_k = \begin{cases} +1 & \text{with probability } p_k(1 - q_k)\langle q \rangle \\ -1 & \text{with probability } p_k q_k(1 - \langle q \rangle) \\ 0 & \text{otherwise} \end{cases}. \quad (6)$$

Note that the updating process of the whole network (not just one node) is a Poisson process of rate $n\lambda$. Hence the increment of m_k in a time interval $(t, t + dt)$ is given by

$$\Delta m_k = \begin{cases} +1 & \text{with probability } n_k(1 - q_k)\langle q \rangle \lambda dt \\ -1 & \text{with probability } n_k q_k(1 - \langle q \rangle) \lambda dt \\ 0 & \text{otherwise} \end{cases}, \quad (7)$$

where we used the fact $n_k = np_k$. We can now calculate the expectation and variance of the random variable Δm_k . Its expectation is given by

$$E[\Delta m_k] = n_k(1 - q_k)\langle q \rangle \lambda dt - n_k q_k(1 - \langle q \rangle) \lambda dt = n_k(\langle q \rangle - q_k) \lambda dt. \quad (8)$$

Its second moment is equal to

$$\begin{aligned} E[(\Delta m_k)^2] &= n_k(1 - q_k)\langle q \rangle \lambda dt + n_k q_k(1 - \langle q \rangle) \lambda dt \\ &= n_k(\langle q \rangle + q_k - 2\langle q \rangle q_k) \lambda dt. \end{aligned} \quad (9)$$

Hence the variance is given by

$$\begin{aligned} \text{var}[\Delta m_k] &= E[(\Delta m_k)^2] - (E[\Delta m_k])^2 \\ &= n_k(\langle q \rangle + q_k - 2\langle q \rangle q_k) \lambda dt + o(dt) \\ &= n_k \sigma_k^2 \lambda dt + o(dt), \end{aligned} \quad (10)$$

where

$$\sigma_k^2 \equiv \langle q \rangle + q_k - 2\langle q \rangle q_k. \quad (11)$$

By definition, $q_k = m_k/n_k$, so we have (to dt order)

$$E[\Delta q_k] = \frac{1}{n_k} E[\Delta m_k] = (\langle q \rangle - q_k) \lambda dt, \quad (12)$$

and

$$\text{var}[\Delta q_k] = \frac{1}{n_k^2} \text{var}[\Delta m_k] = \frac{1}{n_k} \sigma_k^2 \lambda dt. \quad (13)$$

The increment step of Δq_k is $1/n_k$. When n is large this step is small, and Eq. (12) and (13) can be approximated by a continuous process described by the following stochastic differential equation

$$dq_k = (\langle q \rangle - q_k) \lambda dt + \frac{1}{\sqrt{n_k}} \sigma_k \sqrt{\lambda} dB_t^{(k)}, \quad (14)$$

where $B_t^{(k)}$ are k independent Brownian motions. From now on we redefine the time unit so that $\lambda = 1$. Then Eq. (14) becomes

$$dq_k = (\langle q \rangle - q_k) dt + \frac{1}{\sqrt{n_k}} \sigma_k dB_t^{(k)}. \quad (15)$$

which is the set of equations that governs the dynamics of the social network.

2.3 The solution

2.3.1 Martingale

The quantities q_k and $\langle q \rangle$ in Eq. (15) are all random variables, and σ_k is nonlinear in q_k . As a result Eq. (15) is very hard to solve. However, observe that if we take the weighted average (see Eq. (3)) of both sides of Eq. (15), we obtain

$$d\langle q \rangle = \left\langle \frac{1}{\sqrt{n_k}} \sigma_k dB_t^{(k)} \right\rangle, \quad (16)$$

or

$$\langle q(t) \rangle = \int_0^t \left\langle \frac{1}{\sqrt{n_k}} \sigma_k dB_s^{(k)} \right\rangle = \left(\sum_k k p_k \right)^{-1} \sum_k \frac{k p_k}{\sqrt{n_k}} \int_0^t \sigma_k dB_s^{(k)}. \quad (17)$$

Because the right hand side does not include the dt term, $\langle q \rangle$ is a martingale. Thus its expectation value does not change with time¹:

$$E[\langle q(t) \rangle] = \text{constant}. \quad (18)$$

Note that $\langle q(t) \rangle$ is a positive martingale bounded by 1. From the continuous-time martingale convergence theorem [24] it follows that $\langle q(t) \rangle$ converges to a stable distribution as $t \rightarrow \infty$. It is easy to check that the only two absorbing states are all black and all white, so the stable distribution is described by $P(\text{all black}) = E[\langle q(0) \rangle]$ and $P(\text{all white}) = 1 - E[\langle q(0) \rangle]$.

2.3.2 The large n limit

When n is large $n_k^{-1/2}$ is small, so that we can neglect the fluctuation term in Eq. (15) and write

$$\frac{dq_k}{dt} = \langle q \rangle - q_k. \quad (19)$$

This amounts to a mean-field approximation. We divide the nodes into different groups according to their degrees, so that all nodes in the same group have the same degree. If when n is large the size n_k of each group is also large, then we can approximately neglect the fluctuations within each group and replace the group-wise random variables m_k , q_k by their mean values. In this sense Eq. (19) can be regarded as a set of normal differential equations which contain deterministic variables only. Since $\langle q \rangle$ is now deterministic, Eq. (18) becomes

$$\langle q \rangle = \text{constant}. \quad (20)$$

¹This conservation law is independently reported in [37].

Thus Eq. (19) can be easily solved. The solution is

$$q_k(t) = q_k(0)e^{-t} + \langle q(0) \rangle (1 - e^{-t}). \quad (21)$$

We see that for each k ,

$$\lim_{t \rightarrow \infty} q_k(t) = \langle q \rangle. \quad (22)$$

Because $q = \sum n_k q_k / \sum n_k$ is a simple average over q_k , we have from Eq. (21)

$$q(t) = q(0)e^{-t} + \langle q(0) \rangle (1 - e^{-t}) \quad (23)$$

and

$$\lim_{t \rightarrow \infty} q(t) = \langle q \rangle. \quad (24)$$

2.3.3 The convergence time

Based on a conservation result similar to that given in Section 2.3.1, Sood and Redner [34] estimated the convergence time of voter models on heterogeneous graphs. They showed that for a network of n nodes with an arbitrary but uncorrelated degree distribution (same as our assumption), the mean time to reach consensus T_n scales as $n\mu_1^2/\mu_2$, where μ_k is the k th moment of the degree distribution. Thus on a regular graph with $O(1)$ degree $T_n \sim n$. On a scale-free graph with degree distribution $p_k \sim k^{-\alpha}$, T_n scales as

$$T_n \sim \begin{cases} n & \alpha > 3, \\ n/\log n & \alpha = 3, \\ n^{(2\alpha-4)/(\alpha-1)} & 2 < \alpha < 3, \\ (\log n)^2 & \alpha = 2, \\ O(1) & \alpha < 2. \end{cases} \quad (25)$$

We see that in most cases T_n diverges as n diverges. As we will also see in Section 2.5, on average each node switches its color many times before the whole system reaches consensus, which means that the convergence time can be so long that in practice a consensus may not even be observed. [7, 37] also provide explicit evidences showing that the characteristic stabilization time diverges with the system size.

2.4 Interpretation of the solution

A direct corollary of Eq. (18) is that if one starts with a nontrivial initial distribution of opinions (i.e., the nodes are not all black or all white), then the long-term stable distribution will not be trivial. This rather surprising

result was tested in a computer experiment described in Section 2.5. In general the overall fraction of black nodes q is not equal to $\langle q \rangle$, so it can change with time. Eq. (24) shows that q approaches $\langle q \rangle$ as time goes on. To put it more clearly, suppose at $t = 0$ the network is colored in some way such that $q \neq \langle q \rangle$, then averagely speaking, as time passes $\langle q \rangle$ stays at its initial value, while q keeps moving towards $\langle q \rangle$. This is also confirmed by simulation. To better compare q and $\langle q \rangle$ we rewrite their definitions as

$$q = \frac{m}{n} = \frac{\sum m_k}{\sum n_k}; \quad (26)$$

$$\langle q \rangle = \frac{\sum k p_k q_k}{\sum k p_k} = \frac{\sum k n_k q_k}{\sum k n_k} = \frac{\sum k m_k}{\sum k n_k}. \quad (27)$$

It becomes clear that in the weighted average $\langle q \rangle$, each node is given a weight k equal to its degree. Thus, Eq. (24) and (27) says that a high-degree node contributes more to the final fraction of colors (decisions) than a low-degree node. Quantitatively, *the contribution of every node is proportional to its degree*. In other words, *high-degree nodes are more influential*. This explains why a relatively small number of people with high social ranks can affect a significant proportion of the whole society in their decision making. We emphasize that our theory *explains, rather than assumes* why high-rank nodes are more influential in affecting opinion formation than low rank nodes. In fact, in our model when a node updates its color, it puts equal weight on all its neighbors. The chance that it will get the color from a high-degree neighbor and the chance that it will get from a low-degree neighbor are the same. However, statistically speaking there are more nodes in the network that are affected by any high-degree node. In other words, people with higher social rank are more influential because *more people pay attention to them* (more people are connected to them). Notice that this not the same as ascribing a higher weight to the single opinion of a high-rank member of the group. Furthermore the fragility of opinion formation that our theory exhibits stems from the possibility that a relatively small number of nodes contribute a significant proportion to the weighted $\langle q \rangle$, thus changing the whole network dramatically. This effect was also tested by computer simulations which we will show in the next section.

2.5 Computer simulations

2.5.1 Regular graph

We performed our first simulation on a regular 20×20 2-dimensional grid. The left edge and the right edge of the grid were connected to each other,

and so were the upper and lower edges. Each node in the graph had degree 4. We randomly assigned 70% of the nodes to be black and 30% to be white. We then randomly picked one node in the network and randomly updated its color to be the color of one of its neighbors. This “pick-and-update” step was repeated 10^4 times so that on average each node got updated 25 times. We then recorded the final q (which equals $\langle q \rangle$ because the graph is regular). We repeated this experiment 100 times, each time on the same network but with a different initial assignment of colors. The average Eq taken over the 100 experiments (each run for 10^4 steps) was 69.72%, which verified the conservation of $E\langle q \rangle$.

It is worth noting that none of the 100 experiments reached consensus after 10^4 steps, showing that it takes a long time for the system to stabilize.

2.5.2 Random colored scale-free network

The results derived in the previous sections apply to arbitrary degree distributions. In order to stress the degree effect, we performed our rest simulations on a connected power-law network of size $n = 10^4$ and $\alpha = 2.7$, whose (continuous) degree distribution is given by $p_k = (\alpha - 1)k^{-\alpha}$, $k \geq 1$. A sample degree distribution for such a network is shown in Fig. 1.

We first created a random network as described and randomly assigned 70% of the nodes to be black and 30% to be white. We then repeated the “pick-and-update” step 10^6 times so that on average each node got updated 100 times, which is a rather large number for a network of this size. These 10^6 steps constitute a “sample path” of the stochastic process, along which both q and $\langle q \rangle$ were calculated as functions of t . We repeated this experiment 100 times, each time on regenerated networks, so that 100 sample paths were collected. Three of those sample paths are shown in Fig. 2 and 3. As can be seen from the figures, $\langle q \rangle$ has a larger variance than q .

It can be also seen from Fig. 2 that even after each node has updated its color 100 times on average, the system still has not yet reached consensus. Therefore we are able to conclude that the characteristic time for the system to stabilize is very long. This is again consistent with the common observation that conformity throughout a society tends to be localized rather than widespread.

If we take the average of $q(t)$ and the $\langle q(t) \rangle$ over all 100 sample paths we get estimates for $Eq(t)$ and $E\langle q(t) \rangle$. These are shown in Fig. 4. It is clear that both Eq and $E\langle q \rangle$ do not change with time, which confirms the prediction of a martingale in Eq. (18).

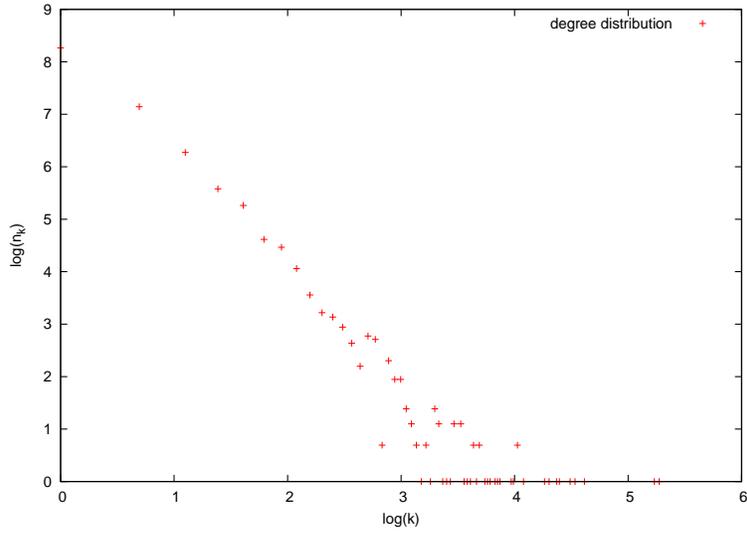


Figure 1: Degree distribution of a network with size 10^4 and $\alpha = 2.7$.

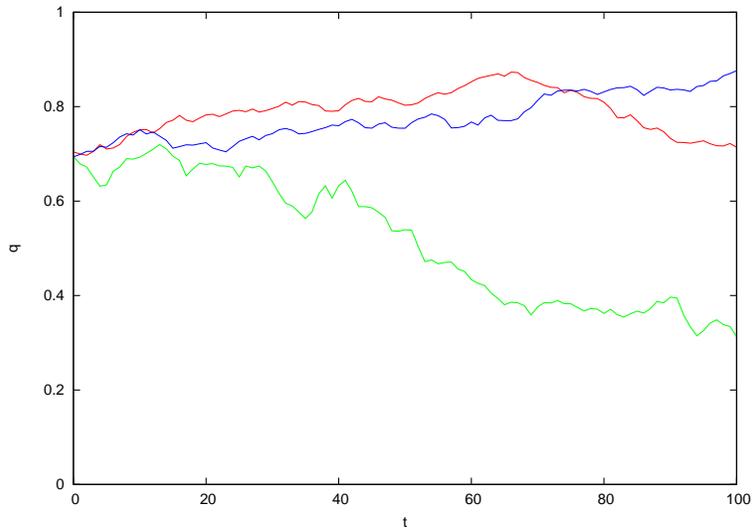


Figure 2: Evolution of the fraction of black nodes, q , on a scale-free random network. The unit of time, t , is 10^4 rounds. The three fraction curves are calculated along three different sample paths, each path sampled on a distinct network. As can be seen none of the three curves reaches 0 or 1 after 100×10^4 rounds, suggesting a long characteristic time of convergence.

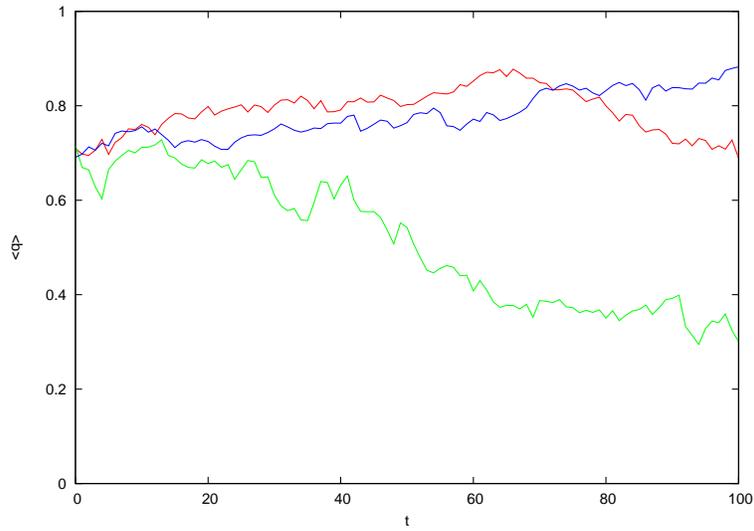


Figure 3: Evolution of the weighted fraction of black nodes, $\langle q \rangle$, on a free random network. The unit of time, t , is 10^4 rounds. The three weighted fraction curves are calculated along the same three sample paths as in Fig. 2.

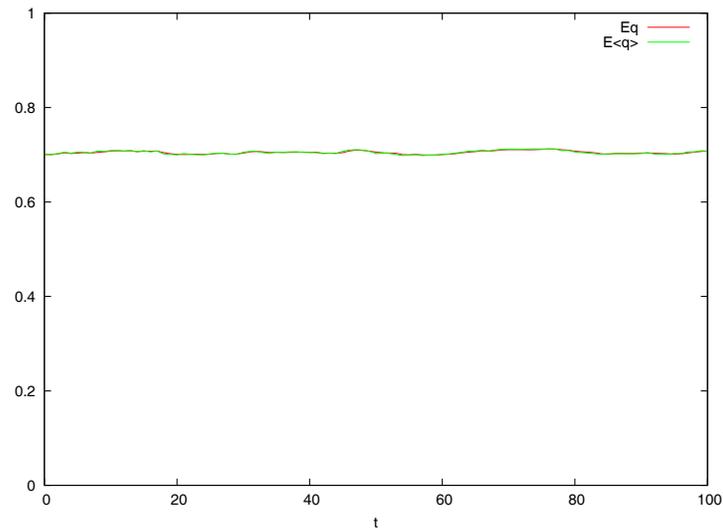


Figure 4: The expected fraction of black nodes (red line) and the expected weighted fraction of black nodes (green line) do not change with time. The expectations are estimated by averaging over 100 sample paths.

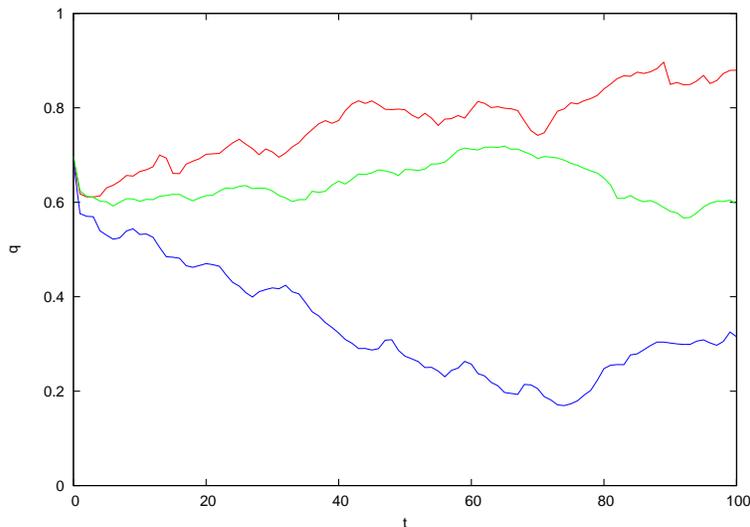


Figure 5: Evolution of the fraction of black nodes, q , on a free network with the 100 highest-degree nodes set to white. The unit of time is 10^4 rounds. The three fraction curves are again calculated along three sample paths on three distinct networks.

2.5.3 Nonrandom color modification

To show that a significant proportion of nodes can be affected by a small number of high-degree nodes, we performed the following experiment. As in Section 2.5.2, we first created a random network, and then randomly assigned 70% of its nodes to be black and 30% to be white. *We then manually assigned the 100 highest-degree nodes to the color white.* Because these 100 nodes constitute only 1% of the whole network and some of them were originally white before the manual assignment, only less than 1% proportion of the network is affected. In other words, the change of q due to the manual step was less than 1%, which can be neglected. On the other hand, because the 100 high-degree nodes contribute a significant weight to the weighted average, the change in the value of $\langle q \rangle$ is significant and cannot be neglected. In fact, $\langle q \rangle$ was lowered from 0.7 to about 0.55 by the color modification. The rest steps remain the same as in Section 2.5.2. We again collected 100 sample paths, three of which are shown in Fig. 5 and 6.

We also take the sample averages of q and $\langle q \rangle$ and plot them as functions of time (Fig. 7). It can be seen that $E\langle q \rangle$ again does not change with time, which further confirms Eq. (18). It is also seen that Eq approaches $E\langle q \rangle$ as

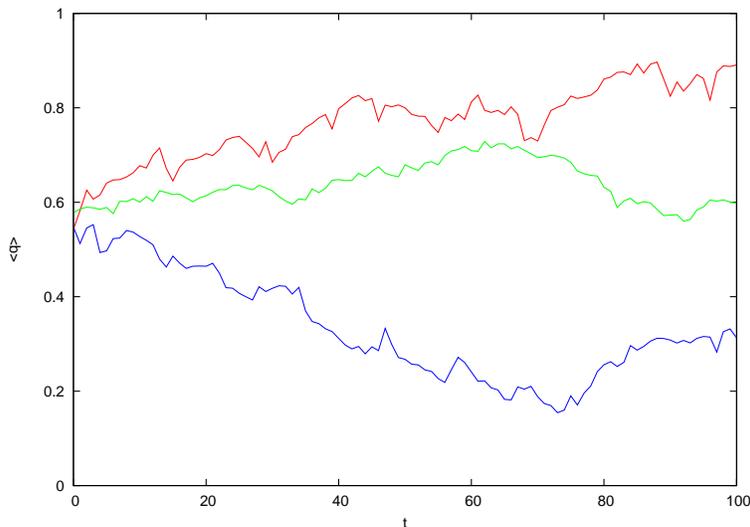


Figure 6: Evolution of the weighted fraction of black nodes, $\langle q \rangle$, on a free network with the 100 highest-degree nodes set to white. The unit of time is 10^4 rounds. The three fraction curves are calculated along the same three sample paths as in Fig. 5.

time goes on, as predicted by Eq. (24).

3 Social networks with a sprinkle of fixed opinions

3.1 Dynamical equations

So far we have assumed that the network is free in the sense that every person-node can change its color at will any number of times. We now extend our model to allow a fraction of the people to have fixed opinions, which translates into nodes with fixed colors. These recalcitrant people or nodes can be regarded as “sources” of the network, in the sense that they can affect others but they themselves cannot be affected by the opinion of others. In a social context, these nodes correspond to “decided” people while the other nodes correspond to “undecided” people. Let b_k be the proportion of degree- k nodes that stay black forever, and let w_k be the proportion of degree- k nodes that stay white forever. The remaining $1 - b_k - w_k$ proportion of degree- k nodes are free to change their colors as before. We now study what the final outcome is going to be for this more realistic case. The

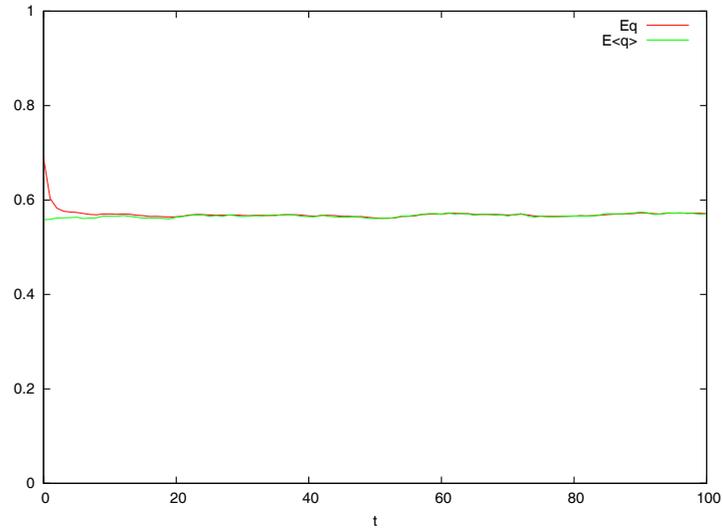


Figure 7: Evolution of the expected value of the fraction of black nodes, Eq , towards the expected weighted fraction $E\langle q \rangle$ as a function of time. At the beginning $Eq = 0.7$ and $E\langle q \rangle = 0.55$. The equilibrium $Eq = E\langle q \rangle = 0.55$ is reached after about 10×10^4 rounds, i.e., after each node updates its color 10 times on average.

difference between a free network and a network with sources is that in the latter case when we randomly choose a node to update, we have to make sure it is free and thus can be updated. Suppose a degree- k node is chosen. At the moment it is chosen, there are $n_k(1 - q_k)$ white nodes with degree k , among which $n_k w_k$ are not free. Therefore the probability that a free white node is chosen is

$$\frac{n_k(1 - q_k) - n_k w_k}{n_k} = 1 - q_k - w_k. \quad (28)$$

Hence we need to replace $1 - q_k$ by $1 - q_k - w_k$ in Eq. (4) to obtain

$$P_{w \rightarrow b}(k) = p_k(1 - q_k - w_k)\langle q \rangle, \quad (29)$$

Similarly, Eq. (5) is modified to

$$P_{b \rightarrow w}(k) = p_k(q_k - b_k)(1 - \langle q \rangle). \quad (30)$$

Repeating the steps in the previous section, we can reach a set of dynamical equations similar to Eq. (15):

$$dq_k = [\langle q \rangle - q_k + b_k(1 - \langle q \rangle) - w_k \langle q \rangle]dt + \frac{1}{\sqrt{n_k}}\sigma_k dB_t^{(k)}, \quad (31)$$

where σ_k is a complicated function of q_k which we do not write out. When $b_k = w_k = 0$ Eq. (31) becomes Eq. (15).

3.2 The solution

Taking the weighted average on both sides of Eq. (31), we have

$$d\langle q \rangle = [b_k(1 - \langle q \rangle) - w_k \langle q \rangle]dt + \left\langle \frac{1}{\sqrt{n_k}}\sigma_k dB_t^{(k)} \right\rangle. \quad (32)$$

Hence $\langle q \rangle$ is no longer a martingale. If we again apply the mean-field approximation to neglect the fluctuation terms, we get

$$\frac{d\langle q \rangle}{dt} = \langle b \rangle(1 - \langle q \rangle) - \langle w \rangle \langle q \rangle. \quad (33)$$

The equilibrium condition is obtained by setting the right hand side equal to zero ($q_\infty = q(t = \infty)$):

$$\langle b \rangle(1 - \langle q_\infty \rangle) - \langle w \rangle \langle q_\infty \rangle = 0, \quad (34)$$

which gives

$$\langle q_\infty \rangle = \frac{\langle b \rangle}{\langle b \rangle + \langle w \rangle}. \quad (35)$$

Therefore as $t \rightarrow \infty$, $\langle q(t) \rangle$ converges to a fixed fraction equal to the weighted proportion of non-free black nodes among all non-free nodes. We see that the final proportion does not depend on the random initial assignment of the colors of the free nodes, although it is possible that the convergence needs such a long time that it can never be reached in reality. Anyway, Eq. (35) shows that the weighted average again plays an important role, indicating that high-degree nodes are more influential to the final outcome.

4 The effect of undecided individuals

4.1 Model

In the first two models we assumed that each person or node can make decisions repeatedly for any number of times. However, in some circumstances, once a node makes a decision it remains unchanged during the whole process of opinion formation. Accordingly, we will now assume that there are two kinds of people or nodes, decided and undecided. A decided node has opinion either black or white, which does not change with time, while an undecided node has no color at the beginning but can obtain one from one of his neighbors after an update of its state. Once it gets a color, it becomes decided and its color stays fixed forever. To conclude, each node has three possible states: black, white and undecided. As before, at each step we randomly pick a node from the network and check its state. If it already has a color (decided), we do nothing. If it is undecided, we randomly pick one of its neighbor. If that neighbor is also undecided, we again do nothing, otherwise we update the first node's color to be the same as its neighbor's.

4.2 Solution

Let b_k and w_k be the proportion of black and white nodes in the network, respectively. We assume that $b_k + w_k < 1$ at $t = 0$ so that there are a finite number of undecided nodes at the beginning. We calculate the probability that the number of k -degree black nodes will be increased by one during an update. For this to happen, first we have to choose an undecided node in step 1, which happens with probability $1 - b_k - w_k$, and then its neighbor we choose in step 2 has to be black, which happens with probability

$\sum k p_k b_k / \sum k p_k$. Thus we have (again neglecting the fluctuation term by mean-field approximation)

$$\frac{db_k}{dt} = (1 - b_k - w_k)\langle b \rangle, \quad (36)$$

and similarly

$$\frac{dw_k}{dt} = (1 - b_k - w_k)\langle w \rangle. \quad (37)$$

Eq. (36) and (37) govern the dynamics of the system. Taking the weighted average of Eq. (36) and (37), we obtain

$$\frac{d\langle b \rangle}{dt} = (1 - \langle b \rangle - \langle w \rangle)\langle b \rangle, \quad (38)$$

and

$$\frac{d\langle w \rangle}{dt} = (1 - \langle b \rangle - \langle w \rangle)\langle w \rangle. \quad (39)$$

To solve Eq. (38) and (39), we take their sum and define $f = 1 - \langle b \rangle - \langle w \rangle$ to get

$$\frac{df}{dt} = f(1 - f). \quad (40)$$

Now f can be solve as

$$f(t) = \frac{1 - f_0}{f_0 e^t + 1 - f_0}, \quad (41)$$

where $f_0 = f(0) = 1 - \langle b(0) \rangle - \langle w(0) \rangle$. Putting this back into Eq. (38) and (39), we can solve out $\langle b \rangle$ and $\langle w \rangle$, which we write down here:

$$\langle b \rangle = \frac{\langle b(0) \rangle e^t}{f_0 e^t + 1 - f_0}, \quad \langle w \rangle = \frac{\langle w(0) \rangle e^t}{f_0 e^t + 1 - f_0}. \quad (42)$$

Hence

$$\frac{\langle b(t) \rangle}{\langle w(t) \rangle} = \frac{\langle b(0) \rangle}{\langle w(0) \rangle} = \text{const.} \quad (43)$$

We see that the weighted black-to-white ratio does not change with time. In fact, this can be seen from Eq. (38) and (39) directly, where the increments of $\langle b \rangle$ and $\langle w \rangle$ is proportional to $\langle b \rangle$ and $\langle w \rangle$, respectively.

5 Information asymmetries

The updating rule of the voter model is *linear* in the sense that the probability that a node will update its color to black is proportional to the number of its black neighbors. This is of course a simplification, as in real world high degree nodes are usually more immune to influence by a single innovating neighbor. A possible extension of our model is to assume a more general updating rule. Such an extension, however, will probably break up the martingale property and we are not going to pursue it here.

Another possible way to extend our model is to incorporate informational asymmetries into our assumptions in such a way that it is possible for A to get information from B but B cannot get information from A . This corresponds to the study of our model on a directed graph and is illustrated in Fig. 8. In this example B, C, D, E can get information from A but A can only get information from D . A directed graph resembles more closely a real life social network, in which low-rank people pay more attention to high-rank people than the other way around. To generalize our model for undirected graphs, from the point of view of our notation we need to do is to replace the numerous appearances of “degree” by “outgoing degree” in Table 1. As an example, p_k now stands for “outgoing degree distribution”. We point out that the outgoing degree distribution of a directed graph can be very different from the degree distribution of the same graph viewed as an undirected graph. For example, node D in Fig. 1 has outgoing degree 2 as a directed graph but degree 3 as an undirected graph. Under the new definition, all our previous results still hold.

6 Discussion

In this paper we presented a theory of opinion formation that explicitly takes into account the structure of the social network in which individuals are embedded. The theory assumes asynchronous choices by individuals among two or three opinions and it predicts the time evolution of the set of opinions from any arbitrary initial condition. We showed that under very general conditions a martingale property ensues, i.e. the expected weighted fraction of the population that holds a given opinion is constant in time. By weighted fraction we mean the fraction of individuals holding a given opinion, averaged over their social connectivity (degree).²

²Note that in the context of epidemic control Dezso and Barabasi established a similar result that it is more efficient to cure the high degree nodes first [10, 11]. However, they

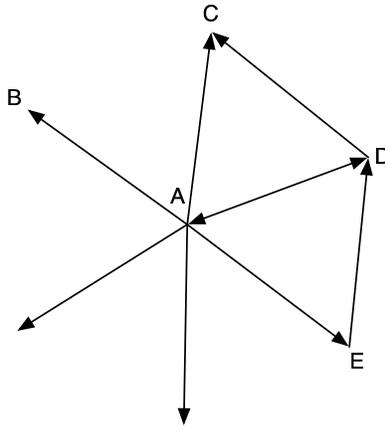


Figure 8: A directed graph. The arrow marks the direction of information flow. For example, A points to C means C can get information from A but not the other way.

Most importantly, this weighted fraction is not either zero or one, but corresponds to a non-trivial distribution in the long time limit. This coexistence of opinions within a social network is in agreement with the often observed locality effect, in which an opinion or a fad is localized to given groups without infecting the *whole* society. Our theory further predicts that a relatively small number of individuals with high social ranks can have a larger effect on opinion formation than individuals with low rank. By high rank we mean people with a large number of social connections. This explains naturally a fragility phenomenon frequently noted within societies, whereby an opinion that seems to be held by a rather large group of people can become nearly extinct in a very short time, a mechanism that is at the heart of fads.

These predictions, which apply to general classes of social networks, including power-law and exponential networks, were verified by computer experiments and extended to the case when some individuals hold fixed opinions throughout the dynamical process. Furthermore, we dealt with the case of information asymmetries, which are characterized by the fact that some individuals are often influenced by other people's opinions while being unable to reciprocate and change their counterpart's views.

did not give a quantitative definition of importance like our proportional relation, nor did they propose any convergence law.

While the assumption of only two or three opinions within a social network may seem restrictive, there are many real world instances where people basically choose among points of view. Examples are the choice among two prevalent technologies [33, 25], elections in two party systems, management fads which consultants and executives need to decide whether to implement or not, and highly polarized attitudes towards government actions in many social settings. Our finding that social structure and ranking do affect the formation of these opinions and that they can coexist with each other are in agreement with many empirical observations. Our findings also cast doubt on the applicability of tipping models to a number of consumer behaviors [21]. While there are clear thresholds in the spread of innovations when network externalities are at play [33, 22, 27] it is not clear that the same phenomenon is observed in situations where externalities are not at play. In most of the consumer behaviors that have been “explained” by tipping point ideas one still observes the coexistence of the old and the new preference or opinions over long times, in contrast with the sudden onset seen in the case of positive externalities.

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